第1章 数値計算の概要

1.1 格子と変数の配置

本モデルでは 図 1.1 に示すように, 水平・鉛直方向の変数をずらして互い違いに配置する. 配置方法は, 水平方向に Arakawa C グリッド, 鉛直方向には Lorenz グリッドを用いている.

すべてのスカラー量 (ϕ: 気圧偏差, 温位偏差, 水蒸気混合比, 雲水混合比, 雨水混合比) を格子の中心に配置する. ベクトル量 (速度 u, w など) は中心から次のように半格子ずらして配置する. x 方向ベクトル量を評価する点は, 鉛直方向にはスカラー量と同じとし水平方向に半格子ずらす. x 方向ベクトル量を評価する点は, 水平方向にはスカラー量と同じとし鉛直方向に半格子ずらす. 格子点のラベルづけおよび境界は 図 1.2 のように設定する.

1.2 空間, 時間方向の離散化の方法

空間方向の離散化は 2 次精度中心差分を用いて行う. 時間方向の離散化は時間分割法を用いて行う. 運動方程式と圧力方程式は短い時間刻み $\Delta\tau$ で時間積分を行う. 音波に関する項の離散化には HE-VI 法を採用し, u の式は前進差分, w, π の式は後退差分で評価する. 音波にかかわらない項についてはリープフロッグ法を用いて積分する. 熱力学の式とその他のトレーサの式は, リープフロッグ法を用いて長い時間刻み Δt で時間積分を行う.

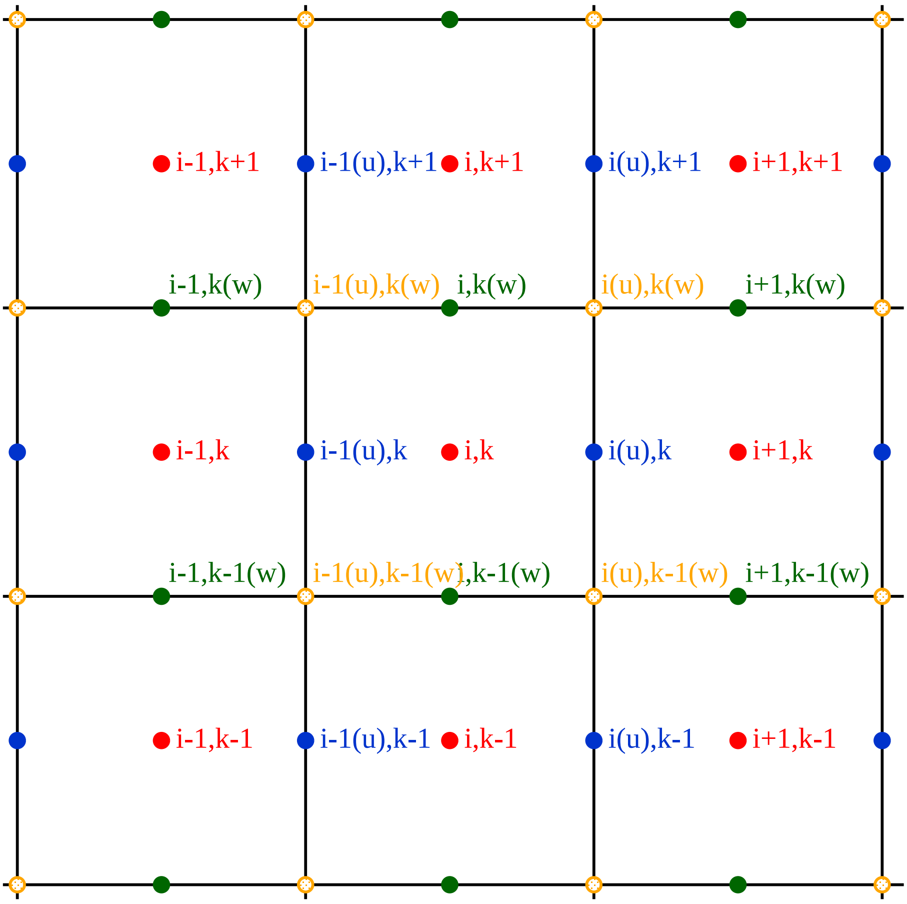

図 1.1: 格子点の配置.

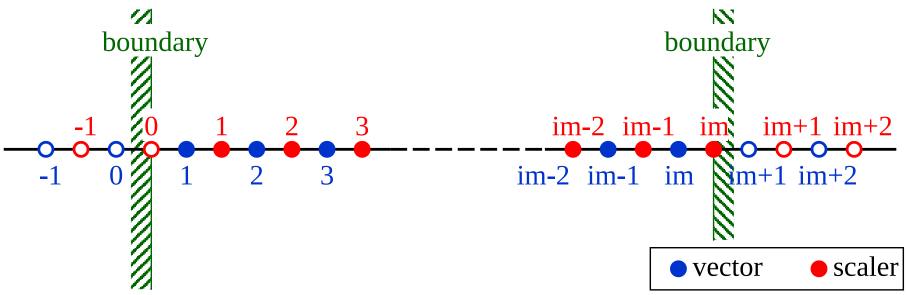

図 1.2: 添字と境界の設定. 実際に計算する添字の範囲は 1 から im とする.

第2章　空間方向の離散化

この節では空間微分の離散化の方法とそのために必要となる平均操作, 境界条件の与え方について説明する. 離散化は 2 次精度差分を用いて行う.

図 1.1 の空間の位置を表す添字として, x 方向フラックスの格子点を $(i(u), k)$, z 方向フラックスの格子点を $(i, k(w))$, スカラー量の格子点を (i, k), 格子の角に当たる点を $(i(u), k(w))$ とする (図 1.1 参照). 但し $1 \le i(u) \le im$, $1 \le i \le im$, $1 \le k(w) \le km$, $1 \le k \le km$ である.

2.1　平均操作

空間微分の離散化を行う前に, そのために必要となる平均操作を定義しておく. 例えば x 方向フラックス格子点で評価される変数をスカラー量の格子点で評価する場合は, フラックス格子点の値を平均してスカラー格子点での値とみなす.

必要となる平均操作を以下に示す. ここでは x 方向のフラックス格子点の変数を $u_{i(u),k}$, z 方向のフラックス格子点の変数を $w_{i,k(w)}$, スカラー格子点の変数を $\pi_{i,k}$ としている.

$$\pi_{i(u),k} \equiv \frac{\pi_{i+1,k} + \pi_{i,k}}{2} \tag{2.1}$$

$$\pi_{i,k(w)} \equiv \frac{\pi_{i,k+1} + \pi_{i,k}}{2} \tag{2.2}$$

$$\pi_{i(u),k(w)} \equiv \frac{\pi_{i,k} + \pi_{i+1,k} + \pi_{i,k+1} + \pi_{i+1,k+1}}{4} \tag{2.3}$$

$$u_{i,k} \equiv \frac{u_{i(u),k} + u_{i-1(u),k}}{2} \tag{2.4}$$

$$u_{i,k(w)} \equiv \frac{u_{i(u),k+1} + u_{i-1(u),k+1} + u_{i(u),k} + u_{i-1(u),k}}{4} \tag{2.5}$$

$$u_{i(u),k(w)} \equiv \frac{u_{i(u),k+1} + u_{i(u),k}}{2} \tag{2.6}$$

$$w_{i,k} \equiv \frac{w_{i,k(w)} + w_{i,k-1(w)}}{2} \tag{2.7}$$

$$w_{i(u),k} \equiv \frac{w_{i+1,k(w)} + w_{i,k(w)} + w_{i+1,k-1(w)} + w_{i,k-1(w)}}{4} \tag{2.8}$$

$$
w_{i(u),k(w)} \equiv \frac{w_{i+1,k(w)} + w_{i,k(w)}}{2} \tag{2.9}
$$

2.2 空間微分の離散化

任意の変数 ϕ の空間微分を 2 次精度差分で離散化すると以下のようになる.

$$
\left[\frac{\partial \phi}{\partial x}\right]_{i,k} \equiv \frac{\phi_{i(u),k} - \phi_{i-1(u),k}}{\Delta x} \tag{2.10}
$$

$$
\left[\frac{\partial \phi}{\partial z}\right]_{i,k} \equiv \frac{\phi_{i,k(w)} - \phi_{i,k-1(w)}}{\Delta z} \tag{2.11}
$$

$$
\left[\frac{\partial \phi}{\partial x}\right]_{i(u),k} \equiv \frac{\phi_{i+1,k} - \phi_{i,k}}{\Delta x} \tag{2.12}
$$

$$
\left[\frac{\partial \phi}{\partial z}\right]_{i(u),k} \equiv \frac{\phi_{i(u),k(w)} - \phi_{i(u),k-1(w)}}{\Delta z} \tag{2.13}
$$

$$
\left[\frac{\partial \phi}{\partial x}\right]_{i,k(w)} \equiv \frac{\phi_{i(u),k(w)} - \phi_{i-1(u),k(w)}}{\Delta x} \tag{2.14}
$$

$$
\left[\frac{\partial \phi}{\partial z}\right]_{i,k(w)} \equiv \frac{\phi_{i,k+1} - \phi_{i,k}}{\Delta z} \tag{2.15}
$$

以上の定義を踏まえて各格子点上におけるスカラー量とフラックス量の空間微分を離散化すると以下のようになる.

スカラー格子点上での空間微分

$$
\left[\frac{\partial \pi}{\partial x}\right]_{i,k} = \frac{1}{2\Delta x}\left(\pi_{i+1,k} - \pi_{i-1,k}\right) \tag{2.16}
$$

$$
\left[\frac{\partial \pi}{\partial z}\right]_{i,k} = \frac{1}{2\Delta z}\left(\pi_{i,k+1} - \pi_{i,k-1}\right) \tag{2.17}
$$

$$
\left[\frac{\partial u}{\partial x}\right]_{i,k} = \frac{1}{\Delta x}\left(u_{i(u),k} - u_{i-1(u),k}\right) \tag{2.18}
$$

$$
\left[\frac{\partial u}{\partial z}\right]_{i,k} = \frac{1}{4\Delta z}\left(u_{i(u),k+1} + u_{i-1(u),k+1} - u_{i(u),k-1} - u_{i-1(u),k-1}\right) \tag{2.19}
$$

$$
\left[\frac{\partial w}{\partial x}\right]_{i,k} = \frac{1}{4\Delta x}\left(w_{i+1,k(w)} + w_{i+1,k-1(w)} - w_{i-1,k(w)} - w_{i-1,k-1(w)}\right) \tag{2.20}
$$

$$\left[\frac{\partial w}{\partial z}\right]_{i,k} = \frac{1}{\Delta z}\left(w_{i,k(w)} - w_{i,k-1(w)}\right) \quad (2.21)$$

$$\left[\frac{\partial \pi u}{\partial z}\right]_{i(u),k} = \frac{1}{\Delta z}\left(\pi_{i(u),k}u_{i(u),k} - \pi_{i-1(u),k}u_{i-1(u),k}\right) \quad (2.22)$$

$$\left[\frac{\partial \pi w}{\partial z}\right]_{i(u),k} = \frac{1}{\Delta z}\left(\pi_{i,k(w)}w_{i,k(w)} - \pi_{i,k-1(w)}w_{i,k-1(w)}\right) \quad (2.23)$$

x 方向フラックス格子点上での空間微分

$$\left[\frac{\partial \pi}{\partial x}\right]_{i(u),k} = \frac{1}{\Delta x}\left(\pi_{i+1,k} - \pi_{i,k}\right) \quad (2.24)$$

$$\left[\frac{\partial \pi}{\partial z}\right]_{i(u),k} = \frac{1}{4\Delta z}\left(\pi_{i+1,k+1} + \pi_{i,k+1} - \pi_{i+1,k-1} - \pi_{i,k-1}\right) \quad (2.25)$$

$$\left[\frac{\partial u}{\partial x}\right]_{i(u),k} = \frac{1}{2\Delta x}\left(u_{i+1(u),k} - u_{i-1(u),k}\right) \quad (2.26)$$

$$\left[\frac{\partial u}{\partial z}\right]_{i(u),k} = \frac{1}{2\Delta z}\left(u_{i(u),k+1} - u_{i(u),k-1}\right) \quad (2.27)$$

$$\left[\frac{\partial w}{\partial x}\right]_{i(u),k} = \frac{1}{2\Delta x}\left(w_{i+1,k(w)} + w_{i+1,k-1(w)} - w_{i,k(w)} - w_{i,k-1(w)}\right) \quad (2.28)$$

$$\left[\frac{\partial w}{\partial z}\right]_{i(u),k} = \frac{1}{2\Delta z}\left(w_{i+1,k(w)} + w_{i,k(w)} - w_{i+1,k-1(w)} - w_{i,k-1(w)}\right) \quad (2.29)$$

$$\left[\frac{\partial^2 u}{\partial x^2}\right]_{i(u),k} = \frac{1}{\Delta x^2}\left(u_{i+1(u),k} - 2u_{i(u),k} + u_{i-1(u),k}\right) \quad (2.30)$$

$$\left[\frac{\partial^2 u}{\partial z^2}\right]_{i(u),k} = \frac{1}{\Delta z^2}\left(u_{i(u),k+1} - 2u_{i(u),k} + u_{i(u),k-1}\right) \quad (2.31)$$

z 方向フラックス格子点上での空間微分

$$\left[\frac{\partial \pi}{\partial x}\right]_{i,k(w)} = \frac{1}{4\Delta x}\left(\pi_{i+1,k} + \pi_{i+1,k+1} - \pi_{i-1,k} - \pi_{i-1,k+1}\right) \quad (2.32)$$

$$\left[\frac{\partial \pi}{\partial z}\right]_{i,k(w)} = \frac{1}{\Delta z}\left(\pi_{i,k+1} - \pi_{i,k}\right) \quad (2.33)$$

$$\left[\frac{\partial u}{\partial x}\right]_{i,k(w)} = \frac{1}{2\Delta x}\left(u_{i(u),k+1} + u_{i(u),k} - u_{i-1(u),k+1(w)} - u_{i-1(u),k}\right) \tag{2.34}$$

$$\left[\frac{\partial u}{\partial z}\right]_{i,k(w)} = \frac{1}{2\Delta z}\left(u_{i(u),k+1} + u_{i-1(u),k+1} - u_{i(u),k} - u_{i-1(u),k}\right) \tag{2.35}$$

$$\left[\frac{\partial w}{\partial x}\right]_{i,k(w)} = \frac{1}{2\Delta x}\left(w_{i+1,k(w)} - w_{i-1,k(w)}\right) \tag{2.36}$$

$$\left[\frac{\partial w}{\partial z}\right]_{i,k(w)} = \frac{1}{2\Delta x}\left(w_{i,k+1(w)} - w_{i,k-1(w)}\right) \tag{2.37}$$

$$\left[\frac{\partial^2 w}{\partial x^2}\right]_{i,k(w)} = \frac{1}{\Delta x^2}\left(w_{i+1,k(w)} - 2w_{i,k(w)} + w_{i-1,k(w)}\right) \tag{2.38}$$

$$\left[\frac{\partial^2 w}{\partial z^2}\right]_{i,k(w)} = \frac{1}{\Delta z^2}\left(w_{i,k+1(w)} - 2w_{i,k(w)} + w_{i,k-1(w)}\right) \tag{2.39}$$

2.3 空間離散化した基礎方程式

静水圧の式：[1]

$$\left[\frac{\partial \overline{\Pi}}{\partial z}\right]_{i,k} = -\frac{g}{c_p \overline{\theta}_{i,k}} \tag{2.40}$$

基本場の密度 $\overline{\rho}_{i,k}$ は以下のように計算する.

$$\overline{\rho}_{i,k} = \frac{p_{0,i,k}}{R_d} \frac{\overline{\Pi}_{i,k}^{c_v/R_d}}{\overline{\theta}_{i,k}} \tag{2.41}$$

運動方程式：

$$\begin{aligned}
\frac{\partial u_{i(u),k}}{\partial t} &= -u_{i(u),k}\left[\frac{\partial u}{\partial x}\right]_{i(u),k} - w_{i(u),k}\left[\frac{\partial u}{\partial z}\right]_{i(u),k} \\
&\quad -c_p \overline{\theta}_{v,i(u),k}\left[\frac{\partial \pi}{\partial x}\right]_{i(u),k} + [D_u]_{i(u),k} \qquad (2.42)\\
\frac{\partial w_{i,k(w)}}{\partial t} &= -u_{i,k(w)}\left[\frac{\partial u}{\partial x}\right]_{i,k(w)} - w_{i,k(w)}\left[\frac{\partial u}{\partial z}\right]_{i,k(w)} \\
&\quad -c_p \overline{\theta}_{v,i,k(w)}\left[\frac{\partial \pi}{\partial z}\right]_{i,k(w)} + g\frac{\theta_{i,k(w)}}{\overline{\theta}_{i,k(w)}} + [D_w]_{i,k(w)}. \qquad (2.43)
\end{aligned}$$

圧力方程式：

$$\frac{\partial \pi_{i,k}}{\partial t} + \frac{\overline{c}_{i,k}^2}{c_p \overline{\rho}_{i,k} \overline{\theta}_{i,k}^2}\left[\frac{\partial \overline{\rho}\overline{\theta}_v u}{\partial x} + \frac{\partial \overline{\rho}\overline{\theta}_v u}{\partial z}\right]_{i,k} = 0. \tag{2.44}$$

基本場の音速 $\overline{c}$ は以下のように計算する.

$$\overline{c}_{i,k}^2 = \frac{c_p R_d}{c_v} \overline{\Pi}_{i,k} \overline{\theta}_{i,k}. \tag{2.45}$$

熱力学の式：

$$\frac{\partial \theta_{i,k}}{\partial t} = -u_{i,k}\left[\frac{\partial \theta}{\partial x}\right]_{i,k} - w_{i,k}\left[\frac{\partial \theta}{\partial z}\right]_{i,k} + [D_\theta]_{i,k}. \tag{2.46}$$

[1] ここには基本場の数値積分方法を記載する.

2.4 境界条件

ここでは離散化した変数に対する境界条件の与え方をまとめる. 考慮する境界条件は, 周期境界条件, 境界ですべりなし条件と応力なし条件である.

2.4.1 周期境界条件の与え方

例として, x 方向フラックス格子点に配置された変数 $u_{i(u),k}$ を考える. 計算領域内の x 方向の添字を $1(u) \sim im(u)$ とし, 糊代部分の格子点数を 2 とする (図 1.2 参照). このとき周期境界条件は以下のように与えられる.

$$u_{0(u),k} = u_{im(u),k} \tag{2.47}$$

$$u_{-1(u),k} = u_{im-1(u),k} \tag{2.48}$$

$$u_{im+1(u),k} = u_{1(u),k} \tag{2.49}$$

$$u_{im+2(u),k} = u_{2(u),k} \tag{2.50}$$

ただし k は任意の整数であり, その範囲は $-1 \leq k \leq km+2$ である.

z 方向フラックス格子点に配置された変数, スカラー格子点に配置された変数に対しても同様に与えることができる.

2.4.2 すべりなし条件の与え方

境界で速度を 0 とする. この場合, 境界をはさんで変数の値が反対称になるように与える.

例として x 方向に境界を与えた場合を考える. x 方向フラックス格子点に配置された変数に対しては.

$$u_{0(u),k} = u_{im(u),k} = 0 \tag{2.51}$$

$$u_{-1(u),k} = -u_{1(u),k} \tag{2.52}$$

$$u_{im+1(u),k} = -u_{im-1(u),k} \tag{2.53}$$

$$u_{im+2(u),k} = -u_{im-2(u),k} \tag{2.54}$$

とする. 境界上に配置されていない変数に対しては,

$$\pi_{0,k} = -\pi_{1,k} \tag{2.55}$$

$$\pi_{-1,k} = -\pi_{1,k} \tag{2.56}$$

$$\pi_{im+1,k} = -\pi_{im-1,k} \tag{2.57}$$

$$\pi_{im+2,k} = -\pi_{im-2,k} \tag{2.58}$$

とする.

2.4.3 応力なし条件の与え方

境界上で法線方向速度を 0, 接線方向速度の法線方向微分を 0 とする. この場合, 境界上で配置された速度成分は境界をはさんで変数の値が反対称になるように与え, 境界上に配置されていない変数に対しては壁をはさんで変数の値が対称になるように与える.

例として x 方向に境界を与えた場合を考える. x 方向フラックス格子点に配置された変数に対しては.

$$u_{0(u),k} = u_{im(u),k} = 0 \tag{2.59}$$

$$u_{-1(u),k} = -u_{1(u),k} \tag{2.60}$$

$$u_{im+1(u),k} = -u_{im-1(u),k} \tag{2.61}$$

$$u_{im+2(u),k} = -u_{im-2(u),k} \tag{2.62}$$

とする. 境界上に配置されていない変数に対しては,

$$\pi_{0,k} = \pi_{1,k} \tag{2.63}$$

$$\pi_{-1,k} = \pi_{1,k} \tag{2.64}$$

$$\pi_{im+1,k} = \pi_{im-1,k} \tag{2.65}$$

$$\pi_{im+2,k} = \pi_{im-2,k} \tag{2.66}$$

とする.

第3章　運動方程式と圧力方程式の時間方向の離散化

空間離散化された運動方程式 (2.42), (2.42) と圧力方程式 (2.44) を時間方向に離散化する．音波に関連する項は短いタイムステップ $\Delta\tau$ で離散化し, その他の項は長いタイムステップ Δt で離散化する．音波に関連する項の離散化には HE-VI 法を採用し, u の式は前進差分, w, π の式は後退差分 (クランク・ニコルソン法) で離散化する．その他の項の離散化にはリープフロッグ法を用いる．離散化した式の計算はまず u の式から行う．得られた $\tau + \Delta\tau$ の u を用いて π を計算し, u, π を用いて w を計算する．

解く方程式を以下のように書き直す.

$$\frac{\partial u_{i(u),k}}{\partial t} = -\left[\bar{c_p}\bar{\theta}_v \frac{\partial(\pi - \alpha Div)}{\partial x}\right]_{i(u),k} + F_{u,i(u),k}, \tag{3.1}$$

$$\frac{\partial w_{i,k(w)}}{\partial t} = -\left[\bar{c_p}\bar{\theta}_v \frac{\partial(\pi - \alpha Div)}{\partial z}\right]_{i,k(w)} + F_{w,i,k(w)}, \tag{3.2}$$

$$\frac{\partial \pi_{i,k}}{\partial t} + \left[\frac{\bar{c}^2}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\frac{\partial(\bar{\rho}\bar{\theta}_v w)}{\partial z}\right]_{i,k} = -\left[\frac{\bar{c}^2}{\bar{c_p}\bar{\theta}_v}\frac{\partial u}{\partial x}\right]_{i,k}. \tag{3.3}$$

ただし u, w の式には音波減衰項 (reference) を加えてある．ここで F_u, F_w は音波に関連しない項,

$$F_{u,i(u),k} = -u_{i(u),k}\left[\frac{\partial u}{\partial x}\right]_{i(u),k} - w_{i(u),k}\left[\frac{\partial u}{\partial z}\right]_{i(u),k} + [D_u]_{i(u),k} \tag{3.4}$$

$$\begin{aligned} F_{w,i,k(w)} &= -u_{i,k(w)}\left[\frac{\partial u}{\partial x}\right]_{i,k(w)} - w_{i,k(w)}\left[\frac{\partial u}{\partial z}\right]_{i,k(w)} \\ &\quad + g\frac{\theta_{i,k(w)}}{\bar{\theta}_{i,k(w)}} + [D_w]_{i,k(w)}. \end{aligned} \tag{3.5}$$

であり, 時刻 t の値で評価する.

3.1 u の式の時間方向の離散化

(3.1) を時間方向に離散化すると以下のようになる.

$$u_{i(u),k}^{\tau+\Delta\tau} = u_{i(u),k}^{\tau} - \left[\bar{c_p}\bar{\theta}_v \Delta\tau \left\{ \frac{\partial \pi^\tau}{\partial x} - \frac{\partial (\alpha Div)^\tau}{\partial x} \right\}\right]_{i(u),k} + F_{u,i(u),k}^{t} \Delta\tau \qquad (3.6)$$

3.2 w, π の式の時間方向の離散化

HE-VI 法を用いるので, w と π の式を連立して解く. w の式において音波減衰項は前進差分, 圧力項は後退差分で離散化する. π の式において水平微分項は (3.6) で求めた $u^{\tau+\Delta\tau}$ を用いて離散化し, 鉛直微分項は後退差分で離散化する.

$$w^{\tau+\Delta\tau} = w^\tau - \bar{c_p}\bar{\theta}_v \Delta\tau \left\{ \beta \frac{\partial \pi^{\tau+\Delta\tau}}{\partial z} + (1-\beta)\frac{\partial \pi^\tau}{\partial z} - \frac{\partial (\alpha Div)^\tau}{\partial z} \right\} + F_w^t \Delta\tau. \qquad (3.7)$$

$$\pi^{\tau+\Delta\tau} + \beta \frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2} \frac{\partial (\bar{\rho}\bar{\theta}_v w^{\tau+\Delta\tau})}{\partial z} = \pi^\tau - (1-\beta)\frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\frac{\partial(\bar{\rho}\bar{\theta}_v w^\tau)}{\partial z} - \frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\theta}_v}\frac{\partial u^{\tau+\Delta\tau}}{\partial x}. \qquad (3.8)$$

ここでは簡単のため格子点位置を表す添字は省略した. (3.8) 式に (3.7) を代入して $w^{\tau+\Delta\tau}$ を消去する.

$$\begin{aligned}
\pi^{\tau+\Delta\tau} &- \beta^2 \frac{\bar{c}^2\Delta\tau^2}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\frac{\partial}{\partial z}\left\{ \left(\bar{c_p}\bar{\rho}\bar{\theta}_v^2\right)\left(\frac{\partial \pi^{\tau+\Delta\tau}}{\partial z}\right)\right\} \\
&= \pi^\tau - (1-\beta)\frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\frac{\partial(\bar{\rho}\bar{\theta}_v w^\tau)}{\partial z} - \frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\theta}_v}\frac{\partial u^{\tau+\Delta\tau}}{\partial x} \\
&\quad -\beta \frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\left[\frac{\partial}{\partial z}\bar{\rho}\bar{\theta}_v \left\{ w^\tau - \bar{c_p}\bar{\theta}_v\Delta\tau \left\{ (1-\beta)\frac{\partial \pi^\tau}{\partial z} - \frac{\partial(\alpha Div)^\tau}{\partial z}\right\} + F_w^t \Delta\tau \right\}\right].
\end{aligned} \qquad (3.9)$$

(3.9) 式右辺を空間方向に離散化し, 格子点位置を表す添字を付けて表すと以下のようになる (計算の詳細は 第 A 節 参照).

$$\begin{aligned}
&\left\{ -\beta^2 \left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\right)_k \frac{1}{\Delta z^2}\left(\bar{c_p}\bar{\rho}\bar{\theta}_v^2\right)_{k(w)} \right\} \pi_{i,k+1}^{\tau+\Delta\tau} \\
&\quad + \left[1 + \beta^2 \left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\right)_k \frac{1}{\Delta z^2}\left\{ \left(\bar{c_p}\bar{\rho}\bar{\theta}_v^2\right)_{k(w)} + \left(\bar{c_p}\bar{\rho}\bar{\theta}_v^2\right)_{k-1(w)} \right\}\right] \pi_{i,k}^{\tau+\Delta\tau} \\
&\quad + \left\{ -\beta^2 \left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\right)_k \frac{1}{\Delta z^2}\left(\bar{c_p}\bar{\rho}\bar{\theta}_v^2\right)_{k-1(w)} \right\} \pi_{i,k-1}^{\tau+\Delta\tau}
\end{aligned}$$

$$
\begin{aligned}
&= \pi^{\tau}_{i,k} - (1-\beta)\left(\frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\right)_k \left\{\frac{\partial(\bar{\rho}\bar{\theta}_v w^{\tau})}{\partial z}\right\}_{i,k} - \left(\frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\theta}_v}\right)_k \left(\frac{\partial u^{\tau+\Delta\tau}}{\partial x}\right)_{i,k} \\
&\quad -\beta\left(\frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\right)_k \left[\frac{\partial}{\partial z}\bar{\rho}\bar{\theta}_v \left\{w^{\tau} - \bar{c_p}\bar{\theta}_v\Delta\tau\left\{(1-\beta)\frac{\partial\pi^{\tau}}{\partial z} - \frac{\partial(\alpha Div)^{\tau}}{\partial z}\right\} + F_w^t\Delta\tau\right\}\right]_{i,k}.
\end{aligned}
\tag{3.10}
$$

3.2.1　境界条件

上下境界を固定壁とする場合, 境界条件は上部下部境界で,

$$
w(i, 0(w)) = 0, \tag{3.11}
$$

$$
w(i, km(w)) = 0 \tag{3.12}
$$

である.

下部境界

下部境界 $(k(w) = 0(w))$ について考える. この時 (3.7) 式は,

$$
\begin{aligned}
\beta\left(\frac{\partial\pi^{\tau+\Delta\tau}}{\partial z}\right)_{i,0(w)} &= \left[\left(\frac{\partial(\alpha Div)^{\tau}}{\partial z}\right) - (1-\beta)\left(\frac{\partial\pi^{\tau}}{\partial z}\right) + \left(\frac{1}{\bar{c_p}\bar{\theta}_v}F_w^t\right)\right]_{i,0(w)} \\
&\equiv E_{i,0(w)}
\end{aligned}
\tag{3.13}
$$

となる. したがって (3.9) 式は以下のようになる.

$$
\begin{aligned}
&\left\{-\beta^2\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\right)_1 \frac{1}{\Delta z^2}\left(\bar{c_p}\bar{\rho}\bar{\theta}_v^2\right)_{1(w)}\right\}\pi^{\tau+\Delta\tau}_{i,2} + \left\{1 + \beta^2\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\right)_1 \frac{1}{\Delta z^2}\left(\bar{c_p}\bar{\rho}\bar{\theta}_v^2\right)_{1(w)}\right\}\pi^{\tau+\Delta\tau}_{i,1} \\
&= \pi^{\tau}_{i,1} - (1-\beta)\left(\frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\right)_1 \left\{\frac{\partial(\bar{\rho}\bar{\theta}_v w^{\tau})}{\partial z}\right\}_{i,1} - \left(\frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\theta}_v}\right)_1 \left(\frac{\partial u^{\tau+\Delta\tau}}{\partial x}\right)_{i,1} \\
&\quad -\beta\left(\frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\right)_1 \left[\frac{\partial}{\partial z}\bar{\rho}\bar{\theta}_v \left\{w^{\tau} - \bar{c_p}\bar{\theta}_v\Delta\tau\left\{(1-\beta)\frac{\partial\pi^{\tau}}{\partial z} - \frac{\partial(\alpha Div)^{\tau}}{\partial z}\right\} + F_w^t\Delta\tau\right\}\right]_{i,1} \\
&\quad -\beta\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\right)_1 \frac{1}{\Delta z}\left(\bar{c_p}\bar{\rho}\bar{\theta}_v^2\right)_{i,0(w)} E_{i,0(w)}.
\end{aligned}
\tag{3.14}
$$

上部境界

上部境界 $(k(w) = km(w))$ について考える. この時 (3.7) 式は,

$$
\beta\left(\frac{\partial\pi^{\tau+\Delta\tau}}{\partial z}\right)_{i,km(w)} = \left[\left(\frac{\partial(\alpha Div)^{\tau}}{\partial z}\right) - (1-\beta)\left(\frac{\partial\pi^{\tau}}{\partial z}\right) + \left(\frac{1}{\bar{c_p}\bar{\theta}_v}F_w^t\right)\right]_{i,km(w)}
$$

$$\equiv \quad E_{i,km(w)} \tag{3.15}$$

となる. したがって (3.9) 式は以下のようになる.

$$\begin{aligned}
&\left\{1+\beta^2\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_{km}\frac{1}{\Delta z^2}\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{km-1(w)}\right\}\pi_{i,km}^{\tau+\Delta\tau} \\
&+\left\{-\beta^2\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_{km}\frac{1}{\Delta z^2}\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{km-1(w)}\right\}\pi_{i,km-1}^{\tau+\Delta\tau} \\
= \;& \pi_{i,km}^{\tau}-(1-\beta)\left(\frac{\bar{c}^2\Delta\tau}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_{km}\left\{\frac{\partial(\bar{\rho}\bar{\theta}_v w^\tau)}{\partial z}\right\}_{i,km}-\left(\frac{\bar{c}^2\Delta\tau}{\bar{c}_p\bar{\theta}_v}\right)_{km}\left(\frac{\partial u^{\tau+\Delta\tau}}{\partial x}\right)_{i,km} \\
&-\beta\left(\frac{\bar{c}^2\Delta\tau}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_{km}\left[\frac{\partial}{\partial z}\bar{\rho}\bar{\theta}_v\left\{w^\tau-\bar{c}_p\bar{\theta}_v\Delta\tau\left\{(1-\beta)\frac{\partial\pi^\tau}{\partial z}-\frac{\partial(\alpha Div)^\tau}{\partial z}\right\}+F_w^t\Delta\tau\right\}\right]_{i,km} \\
&+\frac{\beta}{\Delta z}\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_{km}\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{km(w)}A_{i,km(w)}.
\end{aligned}\tag{3.16}$$

3.2.2 圧力方程式の解き方

(3.10), (3.14), (3.16) 式を連立すると, 以下のような行列式の形式で書くことができる.

$$\begin{pmatrix}
A_1 & B_2 & & 0 \\
C_1 & \ddots & \ddots & \\
 & \ddots & \ddots & B_{km} \\
0 & & C_{km-1} & A_{km}
\end{pmatrix}
\begin{pmatrix}
\pi_{1,1} & \pi_{2,1} & \cdots & \pi_{im,1} \\
\pi_{1,2} & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \vdots \\
\pi_{1,km} & \cdots & \cdots & \pi_{im,km}
\end{pmatrix}^{\tau+\Delta\tau}
=
\begin{pmatrix}
D_{1,1} & D_{2,1} & \cdots & D_{im,1} \\
D_{1,2} & \ddots & \ddots & \vdots \\
\vdots & & \ddots & \vdots \\
D_{1,km} & \cdots & \cdots & D_{im,km}
\end{pmatrix}^{\tau}. \tag{3.17}$$

この連立方程式を解くことで $\pi_{i,k}$ を求める. この連立方程式の係数は以下のように書ける.

$$\begin{aligned}
A_k &= 1+\beta^2\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_k\frac{1}{\Delta z^2}\left\{\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{k(w)}+\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{k-1(w)}\right\} \\
&\quad (k=2,3,\cdots km-1), \\
A_1 &= 1+\beta^2\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_1\frac{1}{\Delta z^2}\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{1(w)}, \\
A_{km} &= 1+\beta^2\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_{km}\frac{1}{\Delta z^2}\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{km-1(w)},
\end{aligned}$$

$$
\begin{aligned}
B_k &= -\beta^2 \left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\right)_{k-1} \frac{1}{\Delta z^2} \left(\bar{c_p}\bar{\rho}\bar{\theta}_v^2\right)_{k-1(w)}, \\
C_k &= -\beta^2 \left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\right)_{k+1} \frac{1}{\Delta z^2} \left(\bar{c_p}\bar{\rho}\bar{\theta}_v^2\right)_{k(w)}, \\
D_{i,k} &= \pi^\tau_{i,k} - (1-\beta)\left(\frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\right)_k \left\{\frac{\partial(\bar{\rho}\bar{\theta}_v w^\tau)}{\partial z}\right\}_{i,k} - \left(\frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\theta}_v}\right)_k \left(\frac{\partial u^{\tau+\Delta\tau}}{\partial x}\right)_{i,k} \\
&\quad -\beta\left(\frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\right)_k \left[\frac{\partial}{\partial z}\bar{\rho}\bar{\theta}_v\left\{w^\tau - \bar{c_p}\bar{\theta}_v\Delta\tau\left\{(1-\beta)\frac{\partial\pi^\tau}{\partial z} - \frac{\partial(\alpha Div)^\tau}{\partial z}\right\} + F^t_w\Delta\tau\right\}\right]_{i,k} \\
&\quad (k = 2, 3, \cdots km-1), \\
D_{i,1} &= \pi^\tau_{i,1} - (1-\beta)\left(\frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\right)_1 \left\{\frac{\partial(\bar{\rho}\bar{\theta}_v w^\tau)}{\partial z}\right\}_{i,1} - \left(\frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\theta}_v}\right)_1 \left(\frac{\partial u^{\tau+\Delta\tau}}{\partial x}\right)_{i,1} \\
&\quad -\beta\left(\frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\right)_1 \left[\frac{\partial}{\partial z}\bar{\rho}\bar{\theta}_v\left\{w^\tau - \bar{c_p}\bar{\theta}_v\Delta\tau\left\{(1-\beta)\frac{\partial\pi^\tau}{\partial z} - \frac{\partial(\alpha Div)^\tau}{\partial z}\right\} + F^t_w\Delta\tau\right\}\right]_{i,1} \\
&\quad -\beta\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\right)_1 \frac{1}{\Delta z}\left(\bar{c_p}\bar{\rho}\bar{\theta}_v^2\right)_{i,0(w)} E_{i,0(w)}, \\
D_{i,km} &= \pi^\tau_{i,km} - (1-\beta)\left(\frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\right)_{km} \left\{\frac{\partial(\bar{\rho}\bar{\theta}_v w^\tau)}{\partial z}\right\}_{i,km} - \left(\frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\theta}_v}\right)_{km} \left(\frac{\partial u^{\tau+\Delta\tau}}{\partial x}\right)_{i,km} \\
&\quad -\beta\left(\frac{\bar{c}^2\Delta\tau}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\right)_{km} \left[\frac{\partial}{\partial z}\bar{\rho}\bar{\theta}_v\left\{w^\tau - \bar{c_p}\bar{\theta}_v\Delta\tau\left\{(1-\beta)\frac{\partial\pi^\tau}{\partial z} - \frac{\partial(\alpha Div)^\tau}{\partial z}\right\} + F^t_w\Delta\tau\right\}\right]_{i,km} \\
&\quad +\beta\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2}\right)_{km} \frac{1}{\Delta z}\left(\bar{c_p}\bar{\rho}\bar{\theta}_v^2\right)_{km(w)} E_{i,km(w)}.
\end{aligned}
$$

ただし,

$$
E_{i,k(w)} = \left(\frac{\partial(\alpha Div)^\tau}{\partial z}\right)_{i,k(w)} - (1-\beta)\left(\frac{\partial\pi^\tau}{\partial z}\right)_{i,k(w)} + \left(\frac{1}{\bar{c_p}\bar{\theta}_v}F^t_w\right)_{i,k(w)}
$$

である.

付 録A　圧力方程式 (3.9) の左辺の空間微分の書き下し

(3.9) 左辺の変形を行う.

$$
\begin{aligned}
(3.9)\,\text{左辺} &= \pi_{i,k}^{\tau+\Delta\tau} \\
&\quad -\beta^2 \left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_k \frac{1}{\Delta z}\left\{\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{k(w)}\left(\frac{\partial\pi^{\tau+\Delta\tau}}{\partial z}\right)_{i,k(w)}\right\} \\
&\quad +\beta^2 \left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_k \frac{1}{\Delta z}\left\{\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{k-1(w)}\left(\frac{\partial\pi^{\tau+\Delta\tau}}{\partial z}\right)_{i,k-1(w)}\right\} \\
&= \pi_{i,k}^{\tau+\Delta\tau} \\
&\quad -\beta^2 \left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_k \frac{1}{\Delta z}\left\{\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{k(w)}\left(\frac{\pi_{i,k+1}^{\tau+\Delta\tau}-\pi_{i,k}^{\tau+\Delta\tau}}{\Delta z}\right)\right\} \\
&\quad +\beta^2 \left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_k \frac{1}{\Delta z}\left\{\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{k-1(w)}\left(\frac{\pi_{i,k}^{\tau+\Delta\tau}-\pi_{i,k-1}^{\tau+\Delta\tau}}{\Delta z}\right)\right\} \\
&= \left\{-\beta^2 \left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_k \frac{1}{\Delta z^2}\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{k(w)}\right\}\pi_{i,k+1}^{\tau+\Delta\tau} \\
&\quad +\left[1+\beta^2 \left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_k \frac{1}{\Delta z^2}\left\{\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{k(w)}+\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{k-1(w)}\right\}\right]\pi_{i,k}^{\tau+\Delta\tau} \\
&\quad +\left\{-\beta^2 \left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_k \frac{1}{\Delta z^2}\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{k-1(w)}\right\}\pi_{i,k-1}^{\tau+\Delta\tau}. \qquad (A.1)
\end{aligned}
$$

下部境界

下部境界 $(k(w) = 0(w))$ について考える. この時 (3.7) 式は,

$$
\begin{aligned}
\beta\left(\frac{\partial\pi^{\tau+\Delta\tau}}{\partial z}\right)_{i,0(w)} &= \left[\left(\frac{\partial(\alpha Div)^\tau}{\partial z}\right)-(1-\beta)\left(\frac{\partial\pi^\tau}{\partial z}\right)+\left(\frac{1}{\bar{c}_p\bar{\theta}_v}F_w^t\right)\right]_{i,0(w)} \\
&\equiv E_{i,0(w)} \qquad (A.2)
\end{aligned}
$$

となるので, (3.9) 式の左辺は, $k=1$ の場合には,

$$
\begin{aligned}
(3.9)\,\text{式左辺} &= \pi_{i,1}^{\tau+\Delta\tau} \\
&\quad -\beta^2\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_1 \frac{1}{\Delta z}\left\{\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{1(w)}\left(\frac{\partial\pi^{\tau+\Delta\tau}}{\partial z}\right)_{i,1(w)}\right\} \\
&\quad +\beta^2\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_1 \frac{1}{\Delta z}\left\{\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{0(w)}\left(\frac{\partial\pi^{\tau+\Delta\tau}}{\partial z}\right)_{i,0(w)}\right\} \\
&= \pi_{i,1}^{\tau+\Delta\tau} \\
&\quad -\beta^2\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_1 \frac{1}{\Delta z}\left\{\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{1(w)}\left(\frac{\pi_{i,2}^{\tau+\Delta\tau}-\pi_{i,1}^{\tau+\Delta\tau}}{\Delta z}\right)\right\} \\
&\quad +\beta^2\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_1 \frac{1}{\Delta z}\left\{\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{0(w)}\left(\frac{\partial\pi^{\tau+\Delta\tau}}{\partial z}\right)_{i,0(w)}\right\} \\
&= \left\{-\beta^2\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_1 \frac{1}{\Delta z^2}\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{1(w)}\right\}\pi_{i,2}^{\tau+\Delta\tau} \\
&\quad +\left\{1+\beta^2\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_1 \frac{1}{\Delta z^2}\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{1(w)}\right\}\pi_{i,1}^{\tau+\Delta\tau} \\
&\quad +\beta\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_1 \frac{1}{\Delta z}\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{0(w)} E_{i,0(w)}
\end{aligned}
$$

上部境界

上部境界 ($k(w)=km(w)$) について考える. (3.9) 式の左辺は,

$$
\begin{aligned}
\beta\left(\frac{\partial\pi^{\tau+\Delta\tau}}{\partial z}\right)_{i,km(w)} &= \left[\left(\frac{\partial(\alpha Div)^\tau}{\partial z}\right)-(1-\beta)\left(\frac{\partial\pi^\tau}{\partial z}\right)+\left(\frac{1}{\bar{c}_p\bar{\theta}_v}F_w^t\right)\right]_{i,km(w)} \\
&\equiv E_{i,km(w)} \qquad \text{(A.3)}
\end{aligned}
$$

となるので, (3.9) 式の左辺は, $k(w)=km(w)$ の場合には,

$$
\begin{aligned}
(3.9)\,\text{式左辺} &= \pi_{i,km}^{\tau+\Delta\tau} \\
&\quad -\beta^2\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_{km} \frac{1}{\Delta z}\left\{\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{km(w)}\left(\frac{\partial\pi^{\tau+\Delta\tau}}{\partial z}\right)_{i,km(w)}\right\} \\
&\quad +\beta^2\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_{km} \frac{1}{\Delta z}\left\{\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{km-1(w)}\left(\frac{\partial\pi^{\tau+\Delta\tau}}{\partial z}\right)_{i,km-1(w)}\right\} \\
&= \pi_{i,km}^{\tau+\Delta\tau} \\
&\quad -\beta\left(\frac{\bar{c}^2\Delta\tau^2}{\bar{c}_p\bar{\rho}\bar{\theta}_v^2}\right)_{km} \frac{1}{\Delta z}\left(\bar{c}_p\bar{\rho}\bar{\theta}_v^2\right)_{km(w)} E_{i,km(w)}
\end{aligned}
$$

$$
\begin{aligned}
& +\beta^2 \left(\frac{\bar{c}^2 \Delta\tau^2}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2} \right)_{km} \frac{1}{\Delta z} \left\{ \left(\bar{c_p}\bar{\rho}\bar{\theta}_v^2 \right)_{km-1(w)} \left(\frac{\pi_{i,km}^{\tau+\Delta\tau} - \pi_{i,km-1}^{\tau+\Delta\tau}}{\Delta z} \right) \right\} \\
= & \left\{ 1 + \beta^2 \left(\frac{\bar{c}^2 \Delta\tau^2}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2} \right)_{km} \frac{1}{\Delta z^2} \left(\bar{c_p}\bar{\rho}\bar{\theta}_v^2 \right)_{km-1(w)} \right\} \pi_{i,km}^{\tau+\Delta\tau} \\
& + \left\{ -\beta^2 \left(\frac{\bar{c}^2 \Delta\tau^2}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2} \right)_{km} \frac{1}{\Delta z^2} \left(\bar{c_p}\bar{\rho}\bar{\theta}_v^2 \right)_{km-1(w)} \right\} \pi_{i,km-1}^{\tau+\Delta\tau} \\
& -\beta \left(\frac{\bar{c}^2 \Delta\tau^2}{\bar{c_p}\bar{\rho}\bar{\theta}_v^2} \right)_{km} \frac{1}{\Delta z} \left(\bar{c_p}\bar{\rho}\bar{\theta}_v^2 \right)_{km(w)} E_{i,km(w)}
\end{aligned}
$$